Journal of Statistical Physics, Vol. 51, Nos. 3/4, 1988

Spontaneous Magnetization of the Ising Model on a Layered Square Lattice

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Received October 14, 1987; revision received January 28, 1988

We have studied the Ising model on a layered square lattice with four different coupling constants and two different magnetic moments. The partition function at zero magnetic field is derived exactly. We propose a formula for the spontaneous magnetization which agrees with the exact low-temperature series expansion up to the 16th order and reduces to the exact result of Au-Yang and McCoy in a special case.

KEY WORDS: Ising model; spontaneous magnetization; layered square lattice; series expansion.

1. INTRODUCTION

The spontaneous magnetization of the Ising model on a rectangular lattice was first announced by Onsager in 1948, although he never published his derivation. Yang[(1)] was the first to publish a derivation of the spontaneous magnetization on a square lattice, and his result was generalized to a rectangular lattice by Chang.[(2)] In 1960, Syozi and Naya[(3)] made a conjecture for the spontaneous magnetization on a generalized square lattice (also called checkerboard lattice). Their conjecture was confirmed recently.[(4–6)]

In 1974, Au-Yang and McCoy[(7)] calculated exactly the spontaneous magnetization on a layered square lattice with three different coupling constants (J_1, J_2, J_2') and the same magnetic moment for all spins. In their model the coupling constant along any horizontal bond is J_1, and the coupling constant along a vertical bond between the jth row and the $(j+1)$th row is J_2 (J_2') if j is even (odd). Their derivation is very complicated. The spontaneous magnetization is obtained as the limiting value of an infinite block Toeplitz determinant. The purpose of the present paper

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0022-4715/88/0500-0491$06.00/0

is to generalize their result to a layered square lattice with four different coupling constants and two different magnetic moments. Unfortunately, mathematical theorems are not available for a general block Toeplitz determinant and we are unable to derive the spontaneous magnetization via block Toeplitz determinant. In this paper we propose a formula for the spontaneous magnetization which agrees with the low-temperature series expansion up to the 16th order.

2. THE LAYERED ISING MODEL

Consider the layered Ising model of N spins on a square lattice with four coupling constants (J_1, J_2, J_1', J_2') and two magnetic moments (m, m') as shown in Fig. 1. Each spin located on the even (odd) rows carries a magnetic moment m (m'). The coupling constant along horizontal bonds on even (odd) rows is J_1 (J_1'). When $J_1 = J_1'$ and $m = m'$, our model reduces to the case of Au-Yang and McCoy.

The partition function at zero magnetic field is

$$Z = \sum_{\sigma = \pm 1} \prod_{\mathrm{nn}} \exp(K_{ij}\sigma_i\sigma_j) \tag{1}$$

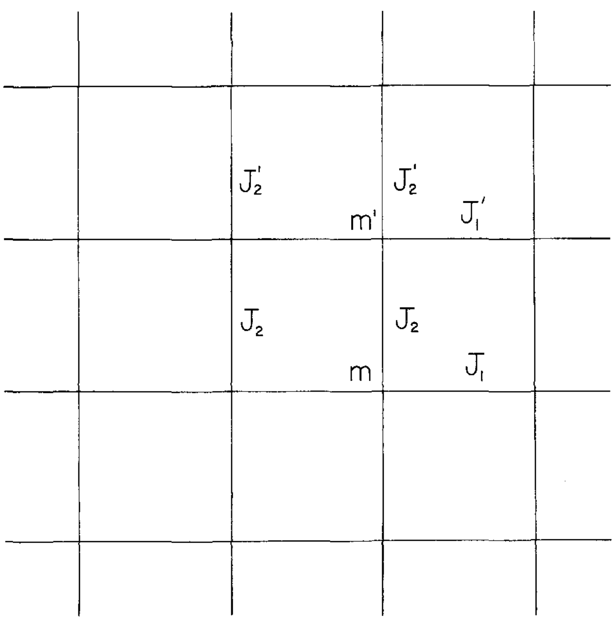

Fig. 1. A layered square lattice.

where $K = J/kT$, J is the coupling constant, σ_i denotes the spin state at lattice site i, and nn means nearest neighbor interaction. The partition function can be derived by the standard method of Pfaffian and dimer city.[8,9] Each vertex is replaced by a dimer city. A unit cell of the dimer lattice (see Fig. 2) corresponds to an eighth-order matrix G with elements

$$g(i, j) = -g^*(j, i) \tag{2}$$

The sign of each element is identified by an arrow such that its pointing from site i to site j implies $\mathrm{sgn}(i, j) = +1$. The matrix elements associated with positive signs are shown explicitly in Fig. 2, except those whose values are unity. We have

$$N^{-1} \log Z = \log 2 + \frac{1}{2} \log(\cosh K_1 \cosh K_2 \cosh K_1' \cosh K_2') + (16\pi^2)^{-1} \iint_0^{2\pi} \log \Delta(\theta, \phi)\, d\theta\, d\phi \tag{3}$$

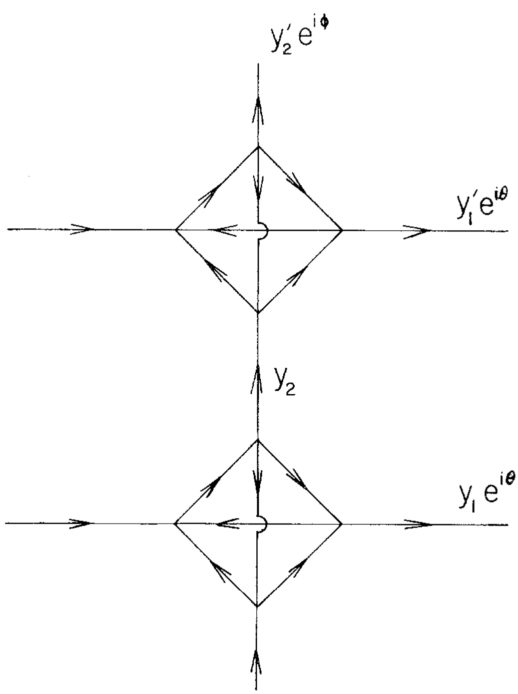

Fig. 2. A unit cell of the dimer lattice.

where $\Delta = \det G$ is the determinant of the 8×8 matrix G:

$$G = \begin{vmatrix} 0 & -1-y_1'e^{-i\theta} & 1 & -1 & 0 & 0 & 0 & 0 \\ 1+y_1'e^{i\theta} & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & y_2'e^{i\phi} \\ 1 & 1 & -1 & 0 & 0 & 0 & -y_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-y_1e^{-i\theta} & 1 & -1 \\ 0 & 0 & 0 & 0 & 1+y_1e^{i\theta} & 0 & -1 & -1 \\ 0 & 0 & 0 & y_2 & -1 & 1 & 0 & 1 \\ 0 & 0 & -y_2'e^{-i\phi} & 0 & 1 & 1 & -1 & 0 \end{vmatrix}$$

After a straightforward calculation, we find

$$\Delta = a + b\cos\theta + c\cos\phi - d\sin^2\theta \tag{4}$$

where

$$\begin{aligned} &a = A^2 + B^2 + C^2 + D^2 \\ &b = 2(AB + CD), \qquad c = 2(AC + BD) \\ &d = 4y_1 y_1'(1 - y_2^2)(1 - y_2'^2), \qquad y_i = \tanh K_i \\ &A = -(1 + y_1 y_1'), \qquad B = y_1 + y_1' \\ &C = y_2 y_2'(1 + y_1 y_1'), \qquad D = y_2 y_2'(y_1 + y_1') \end{aligned}$$

The critical temperature T_c is determined by $A + B + C + D = 0$.

3. SPONTANEOUS MAGNETIZATION

In the special case of $J_1 = J_1'$ and $m = m' = 1$, the spontaneous magnetization M_0 is derived by Au-Yang and McCoy.[7] After a long and difficult calculation, they obtain a remarkably simple result (for $T \leqslant T_c$):

$$\begin{aligned} M_0^8 = {} & (1-c_1^2)(1-c_2^2)(1-c_3^2)(1-c_4^2)(1-zz')^4 \\ & \times [(1-c_1c_2)(1-c_3c_4)(1-z^2)(1-z'^2)]^{-2} \end{aligned} \tag{5}$$

where

$$z = \exp(-2J_2/kT), \qquad z' = \exp(-2J_2'/kT)$$

and $c_j = \exp(i\theta_j)$ are the roots of

$$P_\pm(e^{i\theta}) = a \pm c + b\cos\theta - d\sin^2\theta \tag{6}$$

such that $P_+(c_1) = P_+(c_2) = P_-(c_3) = P_-(c_4) = 0$, $c_j \leqslant 1$.

In the general case, we have calculated the exact low-temperature series expansion[10] for the spontaneous magnetization up to the 16th order.

There are two spontaneous magnetizations associated with the two types of sites. These magnetizations are denoted by $\langle\sigma_{\text{even}}\rangle$ and $\langle\sigma_{\text{odd}}\rangle$, where σ_{even} (σ_{odd}) is the spin on an even (odd) row. We have

$$\langle\sigma_{\text{even}}\rangle = M(x, x', z, z'), \qquad \langle\sigma_{\text{odd}}\rangle = M(x', x, z', z) \tag{7}$$

where

$$x = \exp(-2J_1/kT), \qquad x' = \exp(-2J_1'/kT)$$

$$M(x, x', z, z') = 1 + \sum_{r=2}^{\infty} M_{2r} \tag{8}$$

and

$$\begin{aligned}
M_4 =\ & -2x^2zz' \\
M_6 =\ & -4(xzz')^2 - 2(xx')^2(z^2+z'^2) \\
M_8 =\ & -6x^2(zz')^3 + 4(xx'zz')^2 + 6x^4(zz')^2 \\
& -4(xx')^2(z^4+z'^4) - 2(xx')^2(2x^2+x'^2)\,zz' \\
& -12(xx')^2\,zz'(z^2+z'^2) \\
M_{10} =\ & -4(xx')^4(z^2+z'^2) - 8x^2(zz')^4 - 6(xx')^2(z^6+z'^6) \\
& +4x^4x'^2zz'(z^2+z'^2) - 32(xx'zz')^2(z^2+z'^2) \\
& -20(xx')^2\,zz'(z^4+z'^4) + 24x^2(x^2+x'^2)(zz')^3 \\
& -48x^4(x'zz')^2 - 28x'^4(xzz')^2 \\
M_{12} =\ & -22(xx')^4(z^4+z'^4) - 8(xx')^2(z^8+z'^8) + 60(xzz')^4 \\
& -(xx')^4(6x^2+4x'^2)\,zz' - 10x^2(zz')^5 - 20(x^2zz')^3 \\
& -88(xx')^4\,zz'(z^2+z'^2) - 28(xx')^2\,zz'(z^6+z'^6) \\
& +8x^4x'^2(z^4+z'^4)\,zz' + 12(xx')^2(3x^4+x'^4)(zz')^2 \\
& +16(xx')^4(zz')^2 + 32x^4(x'zz')^2(z^2+z'^2) \\
& -48(xx'zz')^2(z^4+z'^4) - 60(xx')^2(zz')^3(z^2+z'^2) \\
& -276x^4x'^2(zz')^3 - 166x^2x'^4(zz')^3 + 72(xx')^2(zz')^4
\end{aligned}$$

$$
\begin{aligned}
M_{14} = &-10(xx')^2(z^{10}+z'^{10}) - 68(xx')^4(z^6+z'^6) - 12x^2(zz')^6 \\
&-6(xx')^6(z^2+z'^2) + 120(xzz')^4(zz'-x^2) \\
&-36(xx')^2\, zz'(z^8+z'^8) - 336(xx')^4\, zz'(z^4+z'^4) \\
&+4(xx')^4(4x^2+x'^2)\, zz'(z^2+z'^2) - 64(xx'zz')^2(z^6+z'^6) \\
&+12x^4x'^2zz'(z^6+z'^6) - (xx')^4(196x^2+144x'^2)(zz')^2 \\
&-12x^6(x'zz')^2(z^2+z'^2) + 56x^4(x'zz')^2(z^4+z'^4) \\
&-768(xx')^4(zz')^2(z^2+z'^2) + 288(xx')^4(zz')^3 \\
&+176x^2x'^6(zz')^3 - 632x^2(x'zz')^4 + 136(xx')^2(zz')^5 \\
&-96(xx')^2(zz')^4(z^2+z'^2) - 72(xx')^2(zz')^3(z^4+z'^4) \\
&+480x^6x'^2(zz')^3 - 1056x'^2(xzz')^4 \\
&+124x^4x'^2(zz')^3(z^2+z'^2)
\end{aligned}
$$

$$
\begin{aligned}
M_{16} = &-12(xx')^2(z^{12}+z'^{12}) - 156(xx')^4(z^8+z'^8) - 14x^2(zz')^7 \\
&-68(xx')^6(z^4+z'^4) - (xx')^6(8x^2+6x'^2)\, zz' + 210x^4(zz')^6 \\
&-420x^6(zz')^5 + 70(x^2zz')^4 - 44(xx')^2\, zz'(z^{10}+z'^{10}) \\
&-864(xx')^4\, zz'(z^6+z'^6) - 292(xx')^6\, zz'(z^2+z'^2) \\
&+16(xx')^2(x^2-5zz')\, zz'(z^8+z'^8) + 44(xx')^6(zz')^2 \\
&+(xx')^4(60x^2+8x'^2)\, zz'(z^4+z'^4) + 12x^2x'^6zz'(z^6+z'^6) \\
&+54(xx')^4(2x^4+x'^4)(zz')^2 + 80x^4(x'zz')^2(z^6+z'^6) \\
&-2300(xx')^4(zz')^2(z^4+z'^4) - 24x^6(x'zz')^2(z^4+z'^4) \\
&+(xx')^4(336x^2+80x'^2)(zz')^2(z^2+z'^2) + 544(xx'zz')^4 \\
&+(xx')^2(3104x^4+1272x'^4)(zz')^4 + 152x'^6(xzz')^2(z^4+z'^4) \\
&-x^2(64x'^8+2060x^2x'^6+2694x^4x'^4+240x^6x'^2)(zz')^3 \\
&-(xx')^2(1834x'^2+3100x^2)(zz')^5 + 300(xx')^2(zz')^6 \\
&-140(xx')^2(zz')^5(z^2+z'^2) - 128(xx')^2(zz')^4(z^4+z'^4) \\
&-336x'^2(xzz')^4(z^2+z'^2) + 200x^4x'^2(zz')^3(z^4+z'^4) \\
&+(732x^2x'^6-3968x^4x'^4-120x^6x'^2)(zz')^3(z^2+z'^2) \\
&-108(xx')^2(zz')^3(z^6+z'^6)
\end{aligned}
$$

We now propose a formula for M:

$$M(x, x', z, z') = M_0 F(x, x', z, z') \tag{9}$$

where M_0 is defined by (5) and

$$F^4 = N(x, x', z, z')/N(x', x, z, z')$$

with

$$\begin{aligned} N(x, x', z, z') = {} & [(1 - x^2 x'^2)(1 - zz')]^4 \\ & + 4x'^2(1 - x^2 x'^2)^3 zz'(1 - zz')^2 \\ & + (1 - x^2 x'^2)[4(1 - x^2) xx'zz']^2 \\ & - 2[4(1 - x^2)(1 - x'^2) xx'zz']^2 \end{aligned}$$

It can be shown that

$$\begin{aligned} c_j &= [\alpha_+ \pm (\alpha_+^2 - d)^{1/2}][\beta_+ - (\beta_+^2 - d)^{1/2}]/d, \qquad j = 1, 2 \\ &= [\alpha_- - (\alpha_-^2 - d)^{1/2}][\beta_- \pm (\beta_-^2 - d)^{1/2}]/d, \qquad j = 3, 4 \end{aligned} \tag{10}$$

where the upper (lower) signs correspond to $j = 1$ or 3 (2 or 4), and

$$\alpha_\pm = -(A \pm C), \qquad \beta_\pm = B \pm D$$

It follows from expression (10) that

$$\begin{aligned} & 0 \leqslant c_j < 1 \qquad j > 1 \\ & 0 \leqslant c_1 < 1 \qquad \text{if} \quad A + B + C + D > 0 \quad (T_c > T) \\ & c_1 > 1 \qquad \text{if} \quad A + B + C + D < 0 \quad (T_c < T) \end{aligned}$$

$$\begin{aligned} \prod_j (1 - c_j^2) = {} & 16 c_1 c_2 c_3 c_4 \\ & \times d^{-2}(A + B + C + D)(B + C - A - D) \\ & \times (B + D - A - C)(C + D - A - B) \end{aligned} \tag{11}$$

After a straightforward calculation, we get

$$M_0^8 = N(1 - zz')^4/D[(1 - z^2)(1 - z'^2)]^2 \tag{12}$$

where

$$N = [(1 - x^2x'^2)(1 - z^2)(1 - z'^2)]^2 - [4xx'(z + z')(1 + zz')]^2$$

$$\begin{aligned} D = {} & (1 - x^2x'^2)(1 - zz')^4 \\ & + 8zz'(1 - zz')^2[(1 + x^2x'^2)(x^2 + x'^2) - 4x^2x'^2] \\ & + [4zz'(x^2 - x'^2)]^2 \end{aligned}$$

Our conjecture (9) agrees with the exact series expansion (8) up to the 16th order. Notice that

$$M(x, x', z, z') = M(x, x', z', z) \tag{13}$$

which is an exact consequence of the up-down reflection symmetry.

4. EXACTLY SOLUBLE CASES

Case 1. $J_1 = J_1'$. In this case we have $x = x'$. It follows from (13) that

$$\langle \sigma_{\text{even}} \rangle = \langle \sigma_{\text{odd}} \rangle = M(x, x', z, z') \tag{14}$$

Since $F = 1$ and $M = M_0$, our conjecture agrees with the exact result of Au-Yang and McCoy.

Case 2. $J_1 = 0$ or $J_1' = 0$, When $J_1' = 0$, we have $x' = 1$ and

$$F^4 = 1 + 4zz'/(1 - zz')^2 \tag{15}$$

In this case the layered lattice reduces to a decorated square lattice and can be solved exactly.[11] Our formula (15) agrees with the exact result.

Case 3. J_1 or $J_1' = \infty$. When $J_1' = \infty$ (i.e., $x' = 0$) we have

$$F(x, 0, z, z') = [1 + 4x^2zz'/(1 - zz')^2]^{-1/4}$$

and

$$\begin{aligned} \langle \sigma_{\text{odd}} \rangle &= 1 \\ \langle \sigma_{\text{even}} \rangle &= [1 + 4x^2zz'/(1 - zz')^2]^{-1/2} \end{aligned} \tag{16}$$

In this case, our model reduces to a one-dimensional system and the magnetization can be derived exactly for arbitrary magnetic field H,[12]

$$\begin{aligned} \langle \sigma_{\text{odd}} \rangle &= 1 \\ \langle \sigma_{\text{even}} \rangle &= \sinh K\, (\sinh^2 K + x^2)^{-1/2} \end{aligned} \tag{17}$$

where $K \doteq (H + J_2 + J'_2)/kT$. The exact expression (17) reduces to (16) at $H = 0$.

Case 4. J_2 or $J'_2 = \infty$. When $J'_2 = \infty$ (i.e., $z' = 0$), we have $F = 1$ and

$$\langle \sigma_{\text{even}} \rangle = \langle \sigma_{\text{odd}} \rangle = (1 - k^2)^{-1/8} \tag{18}$$

where $k = 4xx'z/(1 - x^2x'^2)(1 - z^2)$. In this case the layered lattice reduces to a rectangular lattice and (18) is exact.

In addition to above soluble cases, it is possible to sum up exactly all terms in the series expansion to fourth in z and z', but to all orders in x and x'. Our conjecture agrees with such exact results.

4. CONCLUSION

We have proposed a formula for the spontaneous magnetization of the Ising model on a layered square lattice with four different coupling constants and two different magnetic moments. This model includes the layered Ising model of Au-Yang and McCoy as a special case. Our conjecture is supported by the following evidence: (1) The spontaneous magnetization drops to zero at the exact critical temperature. (2) Our expression agrees with the exact low-temperature series expansion up to the 16th order. (3) Our result is exact in several special cases.

ACKNOWLEDGMENTS

One of us (K.Y.L.) thanks Dr. H. Au-Yang and Prof. J. Perk for an interesting discussion. This work was supported by the National Science Council of Taiwan.

REFERENCES

1. C. N. Yang, *Phys. Rev.* **85**:808 (1952).
2. C. H. Chang, *Phys. Rev.* **88**:1422 (1952).
3. I. Syozi and S. Naya, *Prog. Theor. Phys.* **23**:374 (1960).
4. K. Y. Lin and J. M. Fang, *Phys. Lett. A* **109**:121 (1985).
5. K. Y. Lin, *J. Stat. Phys.* **49**:269 (1987).
6. R. J. Baxter, *Proc. Soc. Lond. A* **404**:1 (1986).
7. H. Au-Yang and B. McCoy, *Phys. Rev. B* **10**:3885 (1974).
8. C. A. Hurst and H. S. Green, *J. Chem. Phys.* **33**:1059 (1960).
9. P. W. Kasteleyn, *J. Math. Phys.* **4**:287 (1963).
10. C. Domb, *Adv. Phys.* **9**:149 (1960).
11. I. Syozi, in *Phase Transitions and Critical Phenomena*, Vol. 1, C. Domb and M. S. Green, eds. (Academic Press, London, 1972).
12. R. K. Pathria, *Statistical Mechanics* (Pergamon Press, New York, 1972), p. 418.